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Absolute and Relative Approximation with a Singularity

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1. INTRODUCTION

Let f be a real function, continuous and $\neq 0$ on $[-1, 1]$ and let N be an integer $\geqslant 0$. Consider the problem of relative approximation of f by real polynomials $r(x)$ of degree $\leqslant N$, i.e., approximating 1 by $r(x)/f(x)$, uniformly on $[-1, 1]$. This is the same as the problem of approximating f by r in the norm

$$\sup_{-1 \leqslant x \leqslant 1} |1/f(x)| \cdot |f(x) - r(x)|,$$

a special case of the familiar problem of uniform approximation, with a (positive, continuous) weight function, of a continuous function, by polynomials of degree $\leqslant N$, namely, the case where the weight function is the reciprocal of the approximated function.

To get away from that familiar problem, we modify our assumptions by assuming that $f(0) = 0$, while $f(x) \neq 0$ throughout $[-1, 1] \sim \{0\}$. In fact, we shall assume that, for some natural number k, $x^k/f(x)$ is bounded in $[-1, 1] \sim \{0\}$. Given an integer $n \geqslant 0$, one may wish to approximate 1,

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uniformly on $[-1, 1] \sim \{0\}$, by ratios $x^k p(x)/f(x)$, where p is a real polynomial of degree $\leqslant n$. This was studied in [1]. In the present article we undertake a more ambitious project, namely, for every $x \in [-1, 1] \sim \{0\}$, we consider $R_p(x)$, the largest of $|f(x) - x^k p(x)|$, $|1 - \{x^k p(x)/f(x)\}|$, and we study those p which minimize $\|R_p\| = \sup_{x \in [-1,1] \sim \{0\}} R_p(x)$. It is not difficult to see that $\|R_p\| = \max_{x \in [-1,1]} R_p(x)$, where $R_p(0) = \lim_{t \to 0} \sup |1 - \{t^k p(t)/f(t)\}|$. We are concerned with questions of uniqueness and characterization of the minimizing p's. (That such a p exists is quite straightforward, by a standard compactness argument.)

2. Some Definitions

Let $k(\geqslant 1)$ and $n(\geqslant 0)$ be integers, Π_n the set of all real polynomials of degree $\leqslant n$ (including the constant 0), f a real function, continuous in $[-1, 1]$, $f(0) = 0$, $f(x) \neq 0$ throughout $[-1, 1] \sim \{0\}$, and $\sup_{x \in [-1,1] \sim \{0\}} |x^k/f(x)| < \infty$.

We set

$$\mu = \liminf_{x \to 0} x^k/f(x), \qquad M = \limsup_{x \to 0} x^k/f(x).$$

Let $p \in \Pi_n$. We define

$$e_p(x) = f(x) - x^k p(x), \qquad -1 \leqslant x \leqslant 1;$$

$$E_p(x) = 1 - \frac{x^k p(x)}{f(x)}, \qquad 0 < |x| \leqslant 1;$$

$$\|R_p\| = \sup_{-1 \leqslant x \leqslant 1} R_p(x),$$

where, as above,

$$\begin{aligned} R_p(x) &= \max\{|e_p(x)|, |E_p(x)|\} && \text{for} \quad 0 < |x| \leqslant 1, \\ &= \limsup_{t \to 0} |E_p(t)| && \text{for} \quad x = 0. \end{aligned}$$

Observe that

$$\begin{aligned} R_p(0) &= \max\{\limsup_{t \to 0} E_p(t), |\liminf_{t \to 0} E_p(t)|\} \\ &= \max\{\limsup_{t \to 0} E_p(t), -\liminf_{t \to 0} E_p(t)\}, \end{aligned}$$

and so

$$-\|R_p\| \leqslant \liminf_{t \to 0} E_p(t) \leqslant \limsup_{t \to 0} E_p(t) \leqslant \|R_p\|. \tag{1}$$

An $x \in [-1, 1]$ is called a *critical point* iff $R_p(x) = \|R_p\|$. If $x \neq 0$ is a critical point, we set

$$s(x) = \operatorname{sgn}(x^k), \qquad \sigma(x) = \operatorname{sgn} e_p(x), \qquad S(x) = 1.$$

Let 0 be a critical point. We set

$$s(0) = (-1)^k, \qquad \sigma(0) = 1,$$

while $S(0)$ will be defined below as a set consisting of one or two of -1, 0, 1.

Case I. $\mu M \neq 0$. If $\mu M < 0$, $p(0) = 0$, and $\lim_{t\to 0} \sup E_p(t) = \|R_p\|$, set $S(0) = \{0\}$. Otherwise, $S(0)$ is defined by Table I.

Case II. $\mu M = 0$.

If $\mu < 0 = M$, $p(0) < 0$, and
$$-\|R_p\| = \lim_{t\to 0} \inf E_p(t) \leqslant \lim_{t\to 0} \sup E_p(t) < \|R_p\|, \text{ then } S(0) = \{(-1)^k\}.$$

If $\mu < 0 = M$, $p(0) > 0$, and $\lim_{t\to 0} \sup E_p(t) = \|R_p\|$, then $S(0) = \{(-1)^{k+1}\}$.

If $\mu = 0 < M$, $p(0) < 0$, and $\lim_{t\to 0} \sup E_p(t) = \|R_p\|$, then $S(0) = \{(-1)^k\}$.

If $\mu = 0 < M$, $p(0) > 0$, and
$$-\|R_p\| = \lim_{t\to 0} \inf E_p(t) \leqslant \lim_{t\to 0} \sup E_p(t) < \|R_p\|, \text{ then } S(0) = \{(-1)^{k+1}\}.$$

In all other instances in Case II, let $S(0) = \{0\}$.

Every critical point $\neq 0$ is called an *extremum*. Also, 0 is called an extremum iff it is a critical point, and $S(0)$ is a singleton consisting of 1 or -1.

Finally, 0 is called a *determining point* iff it is a critical point, but not an extremum.

TABLE I

	$-\|R_p\| = \lim_{t\to 0} \inf E_p(t)$, $\lim_{t\to 0} \sup E_p(t) < \|R_p\|$	$-\|R_p\| < \lim_{t\to 0} \inf E_p(t)$, $\lim_{t\to 0} \sup E_p(t) = \|R_p\|$	$-\|R_p\| = \lim_{t\to 0} \inf E_p(t)$, $\lim_{t\to 0} \sup E_p(t) = \|R_p\|$
$p(0) < 0$	$S(0) = \{-\operatorname{sgn}[Mf(-1)f(1)]\}$	$S(0) = \{\operatorname{sgn}[\mu f(-1)f(1)]\}$	$S(0) = \{-\operatorname{sgn}[Mf(-1)f(1)], \operatorname{sgn}[\mu f(-1)f(1)]\}$
$p(0) \geqslant 0$	$S(0) = \{-\operatorname{sgn}[\mu f(-1)f(1)]\}$	$S(0) = \{\operatorname{sgn}[Mf(-1)f(1)]\}$	$S(0) = \{-\operatorname{sgn}[\mu f(-1)f(1)], \operatorname{sgn}[Mf(-1)f(1)]\}$

3. Uniqueness and Characterization of Best Approximations

We assume throughout this Section the hypotheses, definitions and notations of Section 2.

Theorem 1. *Suppose 0 is a determining point for some $p^* \in \Pi_n$. Then p^* minimizes $\|R_p\|$ among all $p \in \Pi_n$.*

Proof. Suppose there is $q \in \Pi_n$ with $\|R_q\| < \|R_{p^*}\|$. If $q(0) = p^*(0)$, then, since 0 is a critical point,

$$\begin{aligned}\|R_q\| &\geqslant \max\{\lim_{t\to 0}\sup E_q(t), -\lim_{t\to 0}\inf E_q(t)\}\\ &= \max\{\lim_{t\to 0}\sup E_{p^*}(t), -\lim_{t\to 0}\inf E_{p^*}(t)\}\\ &= \|R_{p^*}\|,\end{aligned}$$

a contradiction. Hence $\delta = q(0) - p^*(0) \neq 0$.

Case I. $\mu M \neq 0$.

Subcase 1. $S(0) = \{0\}$. We have $\mu < 0 < M$, $p^*(0) = 0$, and $\lim_{t\to 0}\sup E_{p^*}(t) = \|R_{p^*}\|$.

If $\delta > 0$, then

$$\|R_q\| \geqslant \lim_{t\to 0}\sup E_q(t) = \lim_{t\to 0}\sup E_{p^*}(t) - \delta\mu > \|R_{p^*}\|.$$

If $\delta < 0$, then

$$\|R_q\| \geqslant \lim_{t\to 0}\sup E_q(t) = \lim_{t\to 0}\sup E_{p^*}(t) - \delta M > \|R_{p^*}\|;$$

in either case we reach a contradiction.

Subcase 2. $S(0) = \{-1, 1\}$. Then $\lim_{t\to 0}\inf E_{p^*}(t) = -\|R_{p^*}\|$, $\lim_{t\to 0}\sup E_{p^*}(t) = \|R_{p^*}\|$, $\operatorname{sgn}\mu = \operatorname{sgn} M$. Observe that the first equality implies that (a) $p^*(0) \neq 0$, (b) if $p^*(0) < 0$, then $\mu < 0$, and (c) if $p^*(0) > 0$, then $M > 0$.

Suppose $q(0) \geqslant 0$. Then (i) $\lim_{t\to 0}\inf E_q(t) = 1 - Mq(0) = 1 - Mp^*(0) - M\delta$, which equals $-\|R_{p^*}\| - M\delta$ if $p^*(0) > 0$, and $\|R_{p^*}\| - M\delta$ if $p^*(0) < 0$; (ii) $\lim_{t\to 0}\sup E_q(t) = 1 - \mu q(0) = 1 - \mu p^*(0) - \mu\delta$, which equals $\|R_{p^*}\| - \mu\delta$ if $p^*(0) > 0$, and $-\|R_{p^*}\| - \mu\delta$ if $p^*(0) < 0$. Hence, if $p^*(0) > 0$, and $M\delta > 0$, then $\lim_{t\to 0}\inf E_q(t) < -\|R_{p^*}\|$; if $p^*(0) > 0$, and $M\delta < 0$, then $\mu\delta < 0$, and $\lim_{t\to 0}\sup E_q(t) > \|R_{p^*}\|$; if $p^*(0) < 0$, then $\delta > 0$, $\mu < 0$, $M\delta < 0$, and, therefore, $\lim_{t\to 0}\sup E_q(t) \geqslant \lim_{t\to 0}\inf E_q(t) > \|R_{p^*}\|$.

By (1), invariably, $\|R_q\| > \|R_{p^*}\|$, a contradiction.

A similar contradiction is obtained if $q(0) < 0$.

Case II. $\mu M = 0$. One proceeds analogously.

THEOREM 2. *$p^* \in \Pi_n$ minimizes $\|R_p\|$ among all $p \in \Pi_n$ iff* (A) *or* (B) *holds.*

(A) *0 is a determining point for p^*.*

In this case, $\|R_{p^*}\| = 1$ if $\mu M \leqslant 0$, $\|R_{p^*}\| = |(M-\mu)/(M+\mu)|$ if $\mu M > 0$, and, in general, $\|R_p\|$ is minimized over Π_n by more than one polynomial.

(B) *There are points $x_1,\ldots, x_{n+2}$ of $[-1, 1]$, all extrema for p^*, such that $-1 \leqslant x_1 < x_2 < \cdots < x_{n+2} \leqslant 1$, and, for that p^*,*

$$s(x_{j+1})\,\sigma(x_{j+1})\,S(x_{j+1}) = -s(x_j)\,\sigma(x_j)\,S(x_j), \qquad j = 1, 2,\ldots, n+1. \quad (2)$$

(Note that each $S(x_r)$ is a singleton, and we use this symbol to denote its unique element).

In this case, p^* is the unique polynomial minimizing $\|R_p\|$ over Π_n.

Proof. Suppose $p^* \in \Pi_n$ minimizes $\|R_p\|$ among all $p \in \Pi_n$, and 0 is not a determining point for p^*. We shall prove the first sentence of (B). Suppose it is false, and take the largest $N \geqslant 1$, call it m, for which there are, for that p^*, extrema $x_1,\ldots, x_N$ with $-1 \leqslant x_1 < x_2 < \cdots < x_N \leqslant 1$, and

$$s(x_{j+1})\,\sigma(x_{j+1})\,S(x_{j+1}) = -s(x_j)\,\sigma(x_j)\,S(x_j)$$

whenever $1 \leqslant j < N$. Then $1 \leqslant m < n+2$. One can show that there are numbers $t_0, t_1,\ldots, t_m$, $-1 = t_0 \leqslant x_1 < t_1 < x_2 < \cdots < t_{m-1} < x_m \leqslant t_m = 1$, so that, if $0 \leqslant j < m$, then there are no extrema ξ, η for p^* in $[t_j, t_{j+1}]$ with $s(\eta)\,\sigma(\eta)\,S(\eta) = -s(\xi)\,\sigma(\xi)\,S(\xi)$. Set

$$\Pi(x) \equiv \prod_{r=1}^{m-1} (x - t_r)$$

(meaning 1 if $m = 1$), and, for every real η,

$$q_\eta(x) \equiv p^*(x) + \eta\Pi(x).$$

Our aim is a real η for which $\|R_{q_\eta}\| < \|R_{p^*}\|$, a contradiction.

Let ε' be one of $1, -1$ so that $s(x_1)\,\sigma(x_1)\,S(x_1) = \operatorname{sgn}[\varepsilon'\Pi(x_1)]$. (Observe that each $\sigma(x_j) \neq 0$, for otherwise, $f(x) \equiv x^k p^*(x)$, and the first sentence of (B) trivially holds.) Clearly,

$$s(x_j)\,\sigma(x_j)\,S(x_j) = \operatorname{sgn}[\varepsilon'\Pi(x_j)] \neq 0, \qquad j = 1, 2,\ldots, m. \quad (3)$$

Case I. $\mu M > 0$.

(a) Suppose $1 \leqslant j \leqslant m$, $x_j \neq 0$. Then

$$\operatorname{sgn} e_{p^*}(x_j) = \operatorname{sgn}|x_j^k \varepsilon' \Pi(x_j)| \neq 0,$$

$$\operatorname{sgn} E_{p^*}(x_j) = \operatorname{sgn}|x_j^k \varepsilon' \Pi(x_j)/f(x_j)|.$$

Hence, if $0 < \varepsilon < 2\,|e_{p^*}(x_j)|x_j^k \Pi(x_j)|^{-1}|$, then

$$\begin{aligned} |e_{q_{\varepsilon\varepsilon'}}(x_j)| &= |e_{p^*}(x_j) - x_j^k \varepsilon\varepsilon' \Pi(x_j)| \\ &= |e_{p^*}(x_j)| \cdot \left| 1 - \frac{x_j^k \varepsilon\varepsilon' \Pi(x_j)}{e_{p^*}(x_j)} \right| < |e_{p^*}(x_j)|, \\ |E_{q_{\varepsilon\varepsilon'}}(x_j)| &= \frac{|e_{q_{\varepsilon\varepsilon'}}(x_j)|}{|f(x_j)|} < |E_{p^*}(x_j)|; \end{aligned}$$

and, consequently,

$$R_{q_{\varepsilon\varepsilon'}}(x_j) < \|R_{p^*}\|.$$

(b) Suppose 0 is one of $x_1, \ldots, x_m$. Then (see Table I) the relations

$$-\|R_{p^*}\| \leqslant \liminf_{t \to 0} E_{p^*}(t), \qquad \limsup_{t \to 0} E_{p^*}(t) \leqslant \|R_{p^*}\| \tag{4}$$

hold with exactly one equality, say equality holds in the first relation. It is easy to see from Table I that $s(0)\,S(0) = -\operatorname{sgn}\mu = -\operatorname{sgn} M$, and, hence, by (3), $\operatorname{sgn}|\varepsilon' \Pi(0)| = -\operatorname{sgn}\mu = -\operatorname{sgn} M$. Let

$$0 < \varepsilon < \frac{\|R_{p^*}\| - \lim_{t \to 0} \sup E_{p^*}(t)}{|\Pi(0)| \max(|\mu|, |M|)}.$$

Then

$$\begin{aligned} \liminf_{t \to 0} E_{q_{\varepsilon\varepsilon'}}(t) &= \liminf_{t \to 0} \left[E_{p^*}(t) - \frac{\varepsilon\varepsilon' t^k \Pi(t)}{f(t)} \right] \\ &\geqslant \liminf_{t \to 0} E_{p^*}(t) - \limsup_{t \to 0} \varepsilon\varepsilon' \Pi(t) \frac{t^k}{f(t)} \\ &= \begin{Bmatrix} -\|R_{p^*}\| - \varepsilon\varepsilon' \Pi(0)\, M & \text{if } M < 0 \\ -\|R_{p^*}\| - \varepsilon\varepsilon' \Pi(0)\, \mu & \text{if } M > 0 \end{Bmatrix} > -\|R_{p^*}\|. \end{aligned}$$

and

$$\limsup_{t\to 0} E_{q_{\varepsilon\varepsilon'}}(t) \leqslant \limsup_{t\to 0} E_{p^*}(t) - \liminf_{t\to 0} \varepsilon\varepsilon'\Pi(t)\frac{t^k}{f(t)}$$

$$= \begin{cases} \limsup_{t\to 0} E_{p^*}(t) - \varepsilon\varepsilon'\Pi(0)\mu & \text{if } \mu < 0 \\ \limsup_{t\to 0} E_{p^*}(t) - \varepsilon\varepsilon'\Pi(0)M & \text{if } \mu > 0 \end{cases} < \|R_{p^*}\|.$$

Hence

$$R_{q_{\varepsilon\varepsilon'}}(0) < \|R_{p^*}\|.$$

One proceeds similarly if equality holds in the second relation (4).

Case II. $\mu M \leqslant 0$. We can, again, find a real η_1 with

$$R_{q_{\eta_1}}(x_j) \leqslant \|R_{p^*}\| \qquad \text{for} \quad j = 1, 2,..., m.$$

Furthermore, by a straightforward, standard argument, we can always

TABLE II

μ, M	$p(0)$	$\limsup_{t\to 0} E_p(t) = \|R_p\|$	$p(0)$	$\liminf_{t\to 0} E_p(t) = -\|R_p\|$
$0 < \mu \leqslant M$	any sgn	Extremum $\Sigma = 1$	>0	Extremum, $\Sigma = -1$
			$\leqslant 0$	Cannot occur
$\mu \leqslant M < 0$	any sgn	Extremum $\Sigma = -1$	$\geqslant 0$	Cannot occur
			<0	Extremum, $\Sigma = 1$
$\mu < 0 < M$	>0	Extremum, $\Sigma = -1$	>0	Extremum, $\Sigma = -1$
	$=0$	Determining point	$=0$	Cannot occur
	<0	Extremum, $\Sigma = 1$	<0	Extremum, $\Sigma = 1$
$\mu = 0 < M$	$\geqslant 0$	Determining point	>0	Extremum iff $\limsup_{t\to 0} E_p(t) < \|R_p\|$ in which case $\Sigma = -1$
	<0	Extremum, $\Sigma = 1$	$\leqslant 0$	Cannot occur
$\mu < 0 = M$	>0	Extremum, $\Sigma = -1$	$\geqslant 0$	Cannot occur
	$\leqslant 0$	Determining point	<0	Extremum iff $\limsup_{t\to 0} E_p(t) < \|R_p\|$ in which case $\Sigma = 1$
$\mu = 0 = M$	any sgn	Determining point	any sgn	Cannot occur

(regardless of sgn(μM)) choose a real η such that $R_{q_\eta}(x) < \|R_{p^*}\|$ throughout $[-1, 1]$, a contradiction.

Conversely, if $p^* \in \Pi_n$ satisfies (A), then, by Theorem 1, p^* minimizes $\|R_p\|$ over Π_n. If $p^* \in \Pi_n$ satisfies (B), and if $\|R_q\| < \|R_{p^*}\|$ for some $q \in \Pi_n$, then examination of $e_{p^*}(x) - e_q(x)$ and $E_{p^*}(x) - E_q(x)$ shows that p^* and q coincide at $n + 1$ points, and, hence, everywhere, contradicting the last inequality.

We omit proof of the remaining statements of Theorem 2.

THEOREM 3. *Let $B(f)$ be the set of all p^* minimizing $\|R_p\|$ over Π_n. If $\mu M < 0$, then* $B(f) = \{p \in \Pi_n : p(x) \equiv 0$, or $\operatorname{sgn} p(x) = \operatorname{sgn}(x^k/f(x))$ *and* $|p(x)| \leqslant |2f(x)/x^k|$ *throughout* $[-1, 1] \sim \{0\}\}$, *and* $\min_{p \in \Pi_n} \|R_p\| = 1$.

If $\mu M > 0$, *then* $B(f) = \{p \in \Pi_n : p(0) = 2/(M + \mu)$ *and* $2\mu f(x) \cdot x^{-k}(M + \mu)^{-1} \leqslant p(x) \leqslant 2Mf(x)x^{-k}(M + \mu)^{-1}$ *throughout* $[-1, 1] \sim \{0\}\}$, *and* $\min_{p \in \Pi_n} \|R_p\| = |(M - \mu)/(M + \mu)|$.

We omit proof.

We conclude with a table (Table II) classifying all possibilities for the point 0, assumed to be a critical point for some $p \in \Pi_n$ (so that $\|R_p\|$ is either $\lim_{t \to 0} \sup E_p(t)$ or $-\lim_{t \to 0} \inf E_p(t)$). In case 0 is an extremum, we give also the value of the product $s(0)\,\sigma(0)\,S(0)$ (see (2)) which we denote by Σ.

REFERENCE

1. A. BACOPOULOS, O. SHISHA AND G. D. TAYLOR, Relative approximation, *J. Approximation Theory* **15** (1975), 356–365.